

WIS/9/05-MAY-DPP

hep-th/0505039

Branes in $L^{(p,q,r)}$

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Abstract

We have found the solution to the back reaction of putting a stack of coincident D3 and D5 branes in $R^{3,1} \times M_6$, where M_6 is constructed from an infinite class of Sasaki-Einstein spaces, $L^{(p,q,r)}$. The non-zero fluxes associated to 2-form potential suggests the presence of a non-contractible 2-cycle in this geometry. The radial part of the warp factor has the usual form and possess the cascading feature. We argue that generically the duals of these SE spaces will have irrational central charges.

1 Introduction

The construction of various 5-dimensional compact manifold is of great importance in string theory especially in the study of field theory. Wrapping D-branes along the cycles of the Calabi-Yau constructed from this along with adding some extra non-compact directions, i.e. putting stacks of coincident D5 branes and D3 branes along both the compact and non-compact directions teaches us a lot about the corresponding dual field theory. The field theoretic observables are possible to calculate from the gravitational quantities due to a masterly duality between these theories, i.e. ADS/CFT correspondence [1], for a review [2]. Otherwise, the computation of the field theoretic quantities like scaling dimension or R-charge of operators and central charges would have been very difficult to do.

Some of the examples, which preserve some non-zero supersymmetry, that have been studied so far are $S^5, T^{(1,1)}$ [3, 4, 6], $Y^{(p,q)}$ [11, 12, 14]¹, and deformation thereof [5, 7]. These manifolds fall into a class called Sasaki-Einstein(SE) type i.e. it admit a Killing spinor. Of-course, the last one was the most general until very recently [9], where an infinite set of completely new non-singular Sasaki-Einstein manifolds with explicit form of the metric is given, which has been dubbed as $L^{(p,q,r)}$, for some choice of p, q, r . $Y^{(p,q)}$ spaces can be obtained from $L^{(p,q,r)}$ as $Y^{(p,q)} = L^{(p-q, p+q, p)}$ [9]. These SE geometries have been obtained by euclideanising along with taking a special limit called “BPS scaling limit” of the five dimensional rotating Kerr-ADS black hole solution given in [8]. This way of generating SE geometries was started out with [10], where these authors were able to generate the $Y^{(p,q)}$ spaces from euclideanising the same five dimensional Kerr-ADS geometry and taking another special limit. It is interesting to note that the $Y^{(p,q)}$ and $L^{(p,q,r)}$ spaces are obtained by taking different limits of the same two parameter family of five-dimensional rotating blackhole solution of [8].

The isometry of $L^{(p,q,r)}$ is $U(1) \times U(1) \times U(1)$ and becomes a smooth non-singular space for $q \geq p > 0$ and $p + q > r > 0$, but the Calabi-Yau that we shall construct has a singularity at $r = 0$. This isometry goes over to $SU(2) \times U(1) \times U(1)$ for $p + q = 2r$ [9]. The $U(1)^3$ isometry is described by certain linear combinations of Killing vectors: $\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \psi}$ and $\frac{\partial}{\partial \phi}$. It is the orbit of the first Killing vector which determine whether the SE space will be a quasi-regular or regular [9]. The former and the later corresponds to the closed and open orbits of the $\frac{\partial}{\partial \tau}$ Killing vector respectively. Its not clear whether this space has the topology of $S^2 \times S^3$ ². In any case, we argue that this geometry do admit a non-contractible two cycle by turning on NSNS fluxes and finding the situation where the periods become non-vanishing, implies the presence of a 2-cycle and support pope’s claim of the topology of this space to be $S^2 \times S^3$.

Even though, we know the explicit form of the metric obtained through this way, as mentioned above, still it would be interesting to derive it from 11-d

¹ p, q and r are coprime integers.

²This is being confirmed to us by Chris Pope that it admits the topology of $S^2 \times S^3$.

supergravity following [12] and to see whether this falls in the class studied there or it lead to a new way of classifying various supergravity solutions.

In this paper we would like to find solutions to Type IIB supergravity constructed on this SE spaces by constructing a cone on this space and putting stacks of D5 branes on the apex of cone along with a stack a D3 branes which are extended along the flat four directions. As expected, the solution should exhibit the cascade feature and which the solution does.

2 The $L^{(p,q,r)}$ geometry

The five dimensional $L^{(p,q,r)}$ geometry is described by the following form of the metric, which we review following [9]

$$ds_5^2 = \frac{1}{\lambda} \left[(d\tau + \sigma)^2 + \frac{\rho^2}{4\Delta_x} dx^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_x}{\rho^2} \left(\frac{\sin^2 \theta}{\alpha} d\phi + \frac{\cos^2 \theta}{\beta} d\psi \right)^2 + \frac{\Delta_\theta \sin^2 2\theta}{4\rho^2} \left(\frac{\alpha - x}{\alpha} d\phi - \frac{\beta - x}{\beta} d\psi \right)^2 \right], \quad (1)$$

where σ is a one form and the form of it and Δ_x, Δ_θ and ρ^2 are given by

$$\begin{aligned} \sigma &= \frac{(\alpha - x)\sin^2 \theta}{\alpha} d\phi + \frac{(\beta - x)\cos^2 \theta}{\beta} d\psi; \quad \rho^2 = \Delta_\theta - x \\ \Delta_x &= x(\alpha - x)(\beta - x) - \mu; \quad \Delta_\theta = \alpha \cos^2 \theta + \beta \sin^2 \theta. \end{aligned} \quad (2)$$

The ranges of the coordinates are $0 < \theta < \pi/2, x_1 < x < x_2, 0 < \phi, \psi < 2\pi$, where x_1 and x_2 are the two lowest roots of $\Delta_x = 0$ equation and τ is a compact coordinate. The parameter μ can be set to any constant value by rescaling α, β, x . Since, we are not going to provide either the exact form of x_1 and x_2 or the period of τ , so we shall stay with μ . In any case, μ do not appear in x_1, x_2 . For $\mu = 0$, one gets the metric of S^5 [9]. However, we do write down the properties of the roots of x_i with $i = 1, 2, 3$, which might be helpful for the computation of x_1, x_2 .

$$\begin{aligned} \sum_{i=1}^3 x_i &= \alpha + \beta; \quad \prod_{i<j} x_i x_j = \alpha\beta; \quad x_1 x_2 x_3 = \mu, \\ x_1 + x_2 &= \frac{1}{\alpha\beta} [x_1^2 + x_2^2 + x_1 x_2 + \alpha\beta]. \end{aligned} \quad (3)$$

A relation between $x_1 - x_2$ will help us to obtain the exact form of x_1, x_2 . Even if we do not know that still we can proceed by assuming that $x_2 - x_1 \equiv \Lambda$, with Λ a real number as x_1, x_2 are real numbers. Then the two roots are

$$\begin{aligned} x_1 &= \frac{1}{6} [2\alpha\beta - 3\Lambda - \sqrt{-12\alpha\beta + 4\alpha^2\beta^2 - 3\Lambda^2}], \\ x_2 &= \frac{1}{6} [2\alpha\beta + 3\Lambda - \sqrt{-12\alpha\beta + 4\alpha^2\beta^2 - 3\Lambda^2}]. \end{aligned} \quad (4)$$

From this there follows a restriction on α, β and Λ that is $4\alpha^2\beta^2 > 12\alpha\beta + 3\Lambda^2$. Which then imply that the volume of this SE spaces are irrational like those found in $Y^{(p,q)}$ case and by ADS/CFT this implies that the central charges associated to the corresponding field theory is also irrational. Of course, for a special case like

$$4\alpha^2\beta^2 = 12\alpha\beta + 3\Lambda^2, \quad (5)$$

one will have rational roots.

The Calabi-Yau that we are interested in is

$$ds_6^2 = dr^2 + r^2 ds_5^2, \quad (6)$$

and the volume of the 5-d geometry $L^{(p,q,r)}$ is given by

$$Vol_5 = \frac{\Delta\tau\pi^2}{4\alpha\beta\lambda^{\frac{5}{2}}}(x_2 - x_1)(\alpha + \beta - x_1 - x_2), \quad (7)$$

where $\Delta\tau$ is the period of τ and $x_3 = \alpha + \beta - x_1 - x_2$. For completeness, let us mention the form of it and is given by

$$x_3 = \frac{2}{6}[3(\alpha + \beta) - 2\alpha\beta + \sqrt{4\alpha^2\beta^2 - 12\alpha\beta - 3\Lambda^2}]. \quad (8)$$

Using the form of x_3 and $x_2 - x_1 = \Lambda$ yields the volume of 5-d SE space as

$$Vol_5 = \frac{\Delta\tau\pi^2\Lambda}{4\alpha\beta\lambda^{\frac{5}{2}}} x_3. \quad (9)$$

Now if the period of τ is anything other than $\frac{2\pi}{x_3}$ then the volume is irrational and hence the central charge, C , of the corresponding dual theory, which is $C \sim \frac{1}{Vol_5}$ also irrational. However, if the period is exactly $1/x_3$ then the volume and central charge are rational. If we assume that the volume (central charge) of $L^{(p,q,r)}$ spaces are rational then its highly unlikely that for some combination of p, q, r one will generate irrational volume (central charges), as $Y^{(p,q)}$ are special cases of $L^{(p,q,r)}$. However, the converse looks more plausible. For the eq.(5), one will have rational volume and hence central charge provided $\Delta\Lambda$ is rational too. In any case, the exact computation of the Chern numbers might give us the exact result.

In order to have a smooth geometry in 5-d, the parameters need to obey [9] $\alpha, \beta > x_2$ with $x_3 > x_2 \geq x_1 > 0$, which according to [9] imply $q \geq p > 0$ and $p + q > r > 0$. It is interesting to note that for $x_1 = x_2$ the volume of the 5-d space vanishes. In order to avoid that we shall however take $x_2 > x_1$, i.e. $\Lambda > 0$.

It is given in [9] that the Ricci tensor of this compact manifold is given by $R_{\mu\nu} = 4\lambda g_{\mu\nu}$. For $\lambda = 1$, one can very easily conclude that the geometry written in (6) is indeed a Calabi-Yau.

3 The solution

The back reaction of D3 and D5 branes onto a space with $R^{3,1} \times M_6$ makes the geometry to take a warped product from [5]. The supergravity solution

when M_6 is $Y^{(p,q)}$ is derived in [15] and for a deformation to this with one free parameter s in [16] and in [17] a nearby space of $Y^{(p,q)}$ has been studied by solving the Lichnerowicz equation and finding the supergravity solution in that geometry.

To find an $\mathcal{N} = 1$ supersymmetry preserving solution, we do not have to show explicitly that the complex 3-form field strength constructed out of 3-form RR and NSNS fields i.e. $G_3 = F_3 - \frac{i}{g_s}H_3$ is a (2,1) form for the geometry whose M_6 part is described by (6). It is known that if G_3 satisfy ISD condition then its enough for that solution will preserve the above mentioned supersymmetry.

The ansatz to the 10-dimensional geometry is

$$ds^2 = h^{-1/2} ds_4^2 + h^{1/2} \left(dr^2 + r^2 [e^{\theta^2} + e^{\tau^2} + e^{x^2} + e^{+2} + e^{-2}] \right). \quad (10)$$

where the warp factor h is a function of several coordinates and the form of it will be determined later and the 1-forms are defined, with $\lambda = 1$, as

$$\begin{aligned} e^x &= \frac{\rho}{2\sqrt{\Delta_x}} dx; & e^\theta &= \frac{\rho}{\sqrt{\Delta_\theta}} d\theta; & e^\tau &= d\tau + \sigma \\ e^+ &= \frac{\sqrt{\Delta_x}}{\rho} \left(\frac{\sin^2 \theta}{\alpha} d\phi + \frac{\cos^2 \theta}{\beta} d\psi \right) \\ e^- &= \frac{\sqrt{\Delta_\theta} \sin 2\theta}{2\rho} \left(\frac{\alpha - x}{\alpha} d\phi - \frac{\beta - x}{\beta} d\psi \right). \end{aligned} \quad (11)$$

The supergravity solution is derived by taking the dilaton to be a constant and the axion is set to zero i.e. *dilaton* = g_s and $C_0 = 0$, respectively.

The two form, B_2 , NSNS potential is assumed to take the form

$$B_2 = g_s M K \ln r \, \omega, \quad (12)$$

where M and K are constants and are related to normalization of flux and integration constant. The form of ω is

$$\omega = \frac{1}{\sqrt{\Delta_x \Delta_\theta} \sin 2\theta} \left(e^x \wedge e^\theta - e^- \wedge e^+ \right). \quad (13)$$

It is easy to see that this two form object possess several interesting properties like: closed, anti-selfdual with respect to x, θ, \pm coordinates and $\omega \wedge J_4 = 0$, where $J_4 = (1/2)d\sigma = e^\theta \wedge e^- - e^x \wedge e^+$.

The form of 3-form RR and NSNS field strengths, which satisfy ISD condition for $G_3 = F_3 - \frac{i}{g_s}H_3$, are

$$F_3 = M K e^\tau \wedge \omega; \quad H_3 = g_s M K \frac{dr}{r} \wedge \omega. \quad (14)$$

The ansatz to the 5-form field strength \tilde{F}_5 is

$$g_s \tilde{F}_5 = dh^{-1} \wedge dx^0 \wedge \dots \wedge dx^3 + \star_{10} \left[dh^{-1} \wedge dx^0 \wedge \dots \wedge dx^3 \right]. \quad (15)$$

The Bianchi identity associated to 3-form fields are satisfied and are very easy to check and the equation of motion associated to the 3-form field strength do not give us the form of the warp factor. The self duality on the five form field strength is automatic. In order to know the dependence of the warp factor on the coordinates one need to solve for either the equation of \tilde{F}_5 or the equation of motion of the metric and we end up the following equation by assuming that the warp factor depends on r, x, θ coordinates i.e. $h = h(r, x, \theta)$.

$$\begin{aligned} r^5 h_r'' + 5r^4 h_r' + \frac{4r^3}{\rho^2} (h_x'' \Delta_x + h_x' \Delta_x') + \frac{r^3}{\rho^2} (h_\theta'' \Delta_\theta + h_\theta' \Delta_\theta') = \\ = -\frac{2g_s^2 M^2 K^2}{r \Delta_x \Delta_\theta} - \frac{r^3}{\rho^2} \cot 2\theta h_\theta' \Delta_\theta, \end{aligned} \quad (16)$$

where h_r' denotes first partial derivative of h with respect to r and similarly for others i.e. prime denotes partial derivative and the number of times it appear says how many times derivative is being taken and the subscript denote coordinate with respect to which its being evaluated. Equation (16) can be brought to a form where the warp factor can be taken as

$$h(r, x, \theta) = \frac{h_1(r) + h_2(x, \theta)}{r^4}, \quad (17)$$

and then using the separation of variable technique we find the solution to $h_1(r)$ as

$$h_1(r) = -\frac{A}{4} \left[\ln(r/r_0) - (C_2/A)r^4 \right]; \quad \text{with } \ln r_0 = 4C_3/A, \quad (18)$$

where C_2, C_3 s are constants of integration and A is the constant that is used in the separation of variable of technique. $C_1 \equiv 2g_s^2 M^2 K^2$.

The equation for $h_2(x, \theta)$ is

$$\begin{aligned} A \Delta_\theta + \left(\frac{C_1}{\Delta_x} - A x \right) - \frac{C_1 x}{\Delta_x \Delta_\theta} + 4(h_{2x}'' \Delta_x + h_{2x}' \Delta_x') + \\ (h_{2\theta}'' \Delta_\theta + h_{2\theta}' \Delta_\theta' + \Delta_\theta h_{2\theta}' \cot 2\theta) = 0. \end{aligned} \quad (19)$$

However, we have not managed to solve this equation.

4 2-cycles

In this section we are not going to follow the approach of [13] to find various supersymmetric 3-cycles. Rather, what we shall do is to find under which situation the integration of B_2 over S^2 becomes non-zero, the purpose of evaluating these fluxes would be to make connection with the field theoretic variables via AdS/CFT correspondence [5], keeping in mind that there are degeneration points in the 5-d SE space [9] and for the presence of 3-cycle follows from the argument that in a Calabi-Yau there exists a nowhere vanishing holomorphic

3-form. The degeneration point is defined as a point in the space of coordinates where the length of the Killing vector vanishes. For $L^{(p,q,r)}$ the degeneration points are at $\theta = 0, \pi/2$ and at $x = x_1, x_2$, the two lowest roots of $\Delta_x = 0$.

Even though it seems that B_2 is diverging at these points but we suggest to evaluate the integral of these form fields cautiously.

The form of the B_2 field is

$$B_2 = \frac{g_s M K l n r}{\sin 2\theta \sqrt{\Delta_x \Delta_\theta}} \left[\frac{\rho^2}{2\sqrt{\Delta_x \Delta_\theta}} dx \wedge d\theta + \frac{\sqrt{\Delta_x \Delta_\theta}}{2\alpha\beta} \sin 2\theta d\psi \wedge d\phi \right]. \quad (20)$$

Now, by doing a gauge transformation we can gauge away the first piece as locally we can write down this term as an exact form, i.e. $d\Lambda$, and left with the second term. It is interesting to note that the integral of B_2 for constant x and θ for the second term becomes a simple number. More importantly, to fix the coordinates x, θ to some value other than the value that it takes at the degeneration point and for these choice one can show that the volume of the 2-cycle to be non-zero. So, the 2-cycle that we define is by fixing x, θ such that $\theta \neq 0, \pi/2$ and $x \neq x_1, x_2$ then integrating over ψ, ϕ , i.e.

$$S^2 = \int d\psi \wedge d\phi \quad \text{for constant } (x, \theta) \text{ with } \theta \neq 0, \pi/2 \text{ and } x \neq x_1, x_2. \quad (21)$$

The volume of this 2-cycle comes out to be

$$Vol_2 = (2\pi)^2 \frac{\sin 2\theta \sqrt{\Delta_x \Delta_\theta}}{\alpha\beta} \quad (22)$$

We know that the Calabi-Yau constructed from this SE spaces admit the topology of $S^2 \times S^3$ [18] and to justify the presence of the non-contractible 3-cycles, we know the CY is supposed to admit a no where vanishing holomorphic $(3,0)$ form means we are guaranteed to have a 3-cycle³.

Note added: As we were finishing this paper there appeared [18] with which there are some overlaps of this work.

5 Acknowledgment

SSP would like to thank O. Aharony for useful discussions and C. Pope for a useful correspondence. The financial help from Feinberg graduate school is gratefully acknowledged.

³We would like to thank Chris Pope for letting us know that this SE spaces do admit the above mentioned topology.

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